A REFINEMENT OF FONTAINE'S MAP FOR PERFECTOID TOWERS AND ITS APPLICATIONS

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1. Introduction

In commutative algebra, many problems remain open only in the mixed characteristic case. The theory of *perfectoid spaces*, introduced by P. Scholze, provides a revolutionary framework that builds a bridge between objects in mixed characteristic and positive characteristic. Originally developed in the context of arithmetic geometry, this theory led to a major breakthrough with Y. André's resolution of the direct summand conjecture, and has since become one of the most powerful tools in commutative algebra in mixed characteristic.

A central operation in perfectoid theory is *tilting*, which associates to a mixed characteristic perfectoid ring A a positive characteristic perfect(oid) ring $A^\flat$. These two rings are linked by the so-called *Fontaine's monoidal map* $\sharp\colon A^\flat \to A$. On the other hand, a major drawback of perfectoid rings is that they are typically non-Noetherian. To overcome this difficulty, S. Ishiro, K. Nakazato, and K. Shimomoto ([INS25]) introduced the notion of *perfectoid towers*, which consist of sequences of rings approximating a perfectoid ring.

With this background in mind, we apply Fontaine's monoidal map to perfectoid towers. As a consequence, we compare the complete integral closedness of a perfectoid tower with that of its tilt. Moreover, we give a new construction of perfectoid towers arising from ramification theory and establish the normality of their tilts. In this talk (and report), we review Fontaine's map and its applications to perfectoid rings, and explain our results on perfectoid towers. This is based on joint work with Shinnosuke Ishiro and Kazuma Shimomoto ([HIS26]).

Throughout this report, we fix a prime number $p > 0$. All rings are assumed to be commutative.

2. Perfectoid rings and Fontaine's map

Let us briefly review applications of Fontaine's map to perfectoid rings (see [CS24, §2.1] and [EHS24] for more details). We say that an $\mathbb{F}_p$-algebra is *perfect* if the Frobenius endomorphism on it is bijective. Perfectoid rings are generalizations of perfect $\mathbb{F}_p$-algebras to mixed characteristic. While there are several equivalent characterizations of perfectoid rings, we adopt the following definition, which is the most convenient for commutative algebraists.

Definition 2.1 (cf. [INS25, Theorem 3.50])**.** We say that a ring A is *perfectoid* if it contains an element ϖ satisfying the following conditions.

(1) $p \in \varpi^p A$, and A is ϖ-adically complete and separated.
(2) The Frobenius endomorphism on $A/\varpi^p A$ induces an isomorphism of rings $A/\varpi A \xrightarrow{\cong} A/\varpi^p A$.
(3) The multiplicative map

$$A_{\varpi\text{-tor}} \to A_{\varpi\text{-tor}}; \quad x \mapsto x^p$$

is bijective, where $A_{\varpi\text{-tor}} := \{x \in A \mid \varpi^n x = 0 \text{ for some } n > 0\}$.

Example 2.2. (1) An $\mathbb{F}_p$-algebra is perfectoid if and only if it is perfect.
(2) Consider a complete regular local ring $(R, \mathfrak{m}, k)$ whose residue field k is a perfect field of characteristic p. By the Cohen structure theorem, we can write $R \cong W(k)[[x_1, \ldots, x_d]]/(p-$

f), where $W(k)$ is the ring of Witt vectors of k and $f = x_1$ or $f \in (x_1, \dots, x_d)^2$. Then the p-adically completion

$$\left(W(k)[x_1^{\frac{1}{p^\infty}}, \dots, x_d^{\frac{1}{p^\infty}}]/(p-f)\right)^\wedge$$

of the ring obtained by adjoining compatible systems of p-power roots of $x_1, \dots, x_d$ is perfectoid. In the case where R is ramified, this result is due to Shimomoto [Shi16].

For other examples, we refer to [BIM19, Example 3.8].

In the rest of this section, let A be a perfectoid ring that is not an $\mathbb{F}_p$-algebra, and take $\varpi \in A$ as in Definition 2.1.

Definition 2.3. We define the *tilt* of A as the projective limit of $\mathbb{F}_p$-algebras

$$A^\flat \overset{\text{def}}{=} \varprojlim_{\text{Frob}} A/pA = \varprojlim(\cdots \xrightarrow{\text{Frob}} A/pA \xrightarrow{\text{Frob}} A/pA).$$

One can check that $A^\flat$ is a perfect $\mathbb{F}_p$-algebra. Thus to a perfectoid ring A we associate a perfect $\mathbb{F}_p$-algebra $A^\flat$. One of the main aims in perfectoid theory is to compare ring-theoretic properties of A with $A^\flat$. However, there is no ring homomorphism between A and $A^\flat$ because they have different characteristics. Despite this, we have the following monoidal map, which is the main object of this talk.

Proposition-Definition 2.4 (J.-M. Fontaine)**.** *The canonical projection $A \to A/pA$ induces an isomorphism of multiplicative monoids*

$$\varprojlim_{x \mapsto x^p} A \xrightarrow{\cong} \varprojlim_{\text{Frob}} A/pA = A^\flat.$$

We define Fontaine's (monoidal) map $\sharp\colon A^\flat \to A$ *as the composite of the inverse of the isomorphism above followed by the 0-th projection $\varprojlim_{x \mapsto x^p} A \to A$.*

Through the map $\sharp$, one can transfer information from A to $A^\flat$. Let us list some fundamental results on the monoidal map.

Proposition 2.5 ([CS24, §2.1])**.** *Let $\varpi^\flat \in A^\flat$ such that $(\varpi^\flat)^\sharp$ is a unit multiple of ϖ.*

(1) *$\sharp\colon A^\flat \to A$ induces an isomorphism of $\mathbb{F}_p$-algebras*

$$A^\flat/(\varpi^\flat)^p A^\flat \xrightarrow{\cong} A/\varpi^p A.$$

In particular, the ideal $(\varpi^\flat)$ is independent of the choice of $\varpi^\flat$.

(2) *$\sharp\colon A^\flat \to A$ induces an isomorphism of abelian groups*

$$(A^\flat)_{\varpi^\flat\text{-tor}} \xrightarrow{\cong} A_{\varpi\text{-tor}}.$$

In particular, A is ϖ-torsion free if and only if $A^\flat$ is $\varpi^\flat$-torsion free.

For a ring extension $R \subset S$, an element $x \in S$ is called *almost integral over* R if the R-subalgebra $R[x]$ of S is contained in a finitely generated R-module. If every element of S is almost integral over R, then we say that R is *completely integrally closed in* S.

Theorem 2.6 ([EHS24, Main Theorem 1.3])**.** *If A is ϖ-torsion free and A is completely integrally closed in $A[\frac{1}{\varpi}]$, then $A^\flat$ is completely integrally closed in $A^\flat[\frac{1}{\varpi^\flat}]$.*

3. Perfectoid towers and Fontaine's map

By a *tower of rings* we mean a sequence of ring homomorphisms

$$\boldsymbol{R} = \{R_i\}_{i \geq 0} = \{R_i, t_i\}_{i \geq 0} = (R_0 \xrightarrow{t_0} R_1 \xrightarrow{t_1} \cdots).$$

We say that a tower of rings $\boldsymbol{R}$ is *perfect* if it is isomorphic to the tower of the form $R \xrightarrow{\text{Frob}} R \xrightarrow{\text{Frob}} \cdots$ with R a reduced $\mathbb{F}_p$-algebra. Perfectoid towers are generalizations of perfect towers in mixed characteristic.

Definition 3.1 ([INS25, Definition 3.21])**.** Let R be a ring, and $I_0 \subset R$ an ideal. We say that a tower of rings $\boldsymbol{R} = \{R_i, t_i\}_{i\geq 0}$ is a *perfectoid tower arising from* (R, I_0) if it satisfies the following conditions.

(a) $R_0 = R$ and $p \in I_0$.
(b) For any $i \geq 0$, the ring homomorphism $\overline{t_i}\colon R_i/I_0R_i \to R_{i+1}/I_0R_{i+1}$ is injective.
(c) For any $i \geq 0$, the Frobenius endomorphism on R_{i+1}/I_0R_{i+1} (uniquely) factors through $\overline{t_i}$ as follows:

$$\begin{array}{ccc} R_{i+1}/I_0R_{i+1} & \xrightarrow{\mathrm{Frob}} & R_{i+1}/I_0R_{i+1} \\ & \overset{F_i}{\searrow} & \uparrow \overline{t_i} \\ & & R_i/I_0R_i \end{array}$$

(d) For any $i \geq 0$, the ring homomorphism $F_i\colon R_{i+1}/I_0R_{i+1} \to R_i/I_0R_i$ is surjective.
(e) For any $i \geq 0$, I_0R_i is contained in the Jacobson radical of R_i.
(f) I_0 is a principal ideal, and R_1 contains a principal ideal I_1 satisfying the following conditions.
  (f-1) $I_1^p = I_0R_1$.
  (f-2) For every $i \geq 0$, $\mathrm{Ker}(F_i) = I_1(R_{i+1}/I_0R_{i+1})$.
(g) For every $i \geq 0$, $I_0(R_i)_{I_0\text{-tor}} = (0)$. Moreover, there exists a (unique) bijection $(F_i)_{\mathrm{tor}}\colon (R_{i+1})_{I_0\text{-tor}} \to (R_i)_{I_0\text{-tor}}$ such that the following diagram commutes.

$$\begin{array}{ccccc} (R_{i+1})_{I_0\text{-tor}} & \hookrightarrow & R_{i+1} & \twoheadrightarrow & R_{i+1}/I_0R_{i+1} \\ {\scriptstyle (F_i)_{\mathrm{tor}}}\downarrow & & & & \downarrow {\scriptstyle F_i} \\ (R_i)_{I_0\text{-tor}} & \hookrightarrow & R_i & \twoheadrightarrow & R_i/I_0R_i \end{array}$$

Example 3.2. (1) A tower of $\mathbb{F}_p$-algebras is a perfectoid tower arising from some pair $(R, (0))$ if and only if it is a perfect tower.
(2) If $R = W(k)[[x_1, \ldots, x_d]]/(p - f)$ is a complete regular local ring as in Example 2.2 (2), then

$$R \hookrightarrow W(k)[[x_1^{\frac{1}{p}}, \ldots, x_d^{\frac{1}{p}}]]/(p-f) \hookrightarrow \cdots \hookrightarrow W(k)[[x_1^{\frac{1}{p^i}}, \ldots, x_d^{\frac{1}{p^i}}]]/(p-f) \hookrightarrow \cdots$$

is a perfectoid tower arising from $(R, (p))$.

There are many other examples of perfectoid towers; see [INS25], [Ish24], and [IS25].

In the rest of this section, let $\boldsymbol{R} = \{R_i, t_i\}_{i\geq 0}$ be a perfectoid tower arising from some pair (R, I_0). Then the I_0-adically completion $\widehat{R_\infty}$ of the inductive limit $R_\infty := \varinjlim_{i\geq 0} R_i$ is a perfectoid ring ([INS25, Corollary 3.52]). We have the notion of tilts for perfectoid towers:

Definition 3.3 ([INS25, Definition 3.29 and 3.34])**.** For any $i \geq 0$, we define the *i-th small tilt* of $\boldsymbol{R}$ as the projective limit of $\mathbb{F}_p$-algebras

$$R_i^{s.\flat} \overset{\mathrm{def}}{=} \varprojlim \left(\cdots \xrightarrow{F_{i+1}} R_{i+1}/I_0R_{i+1} \xrightarrow{F_i} R_i/I_0R_i \right)$$

The transition maps t_{i+j} for $j \geq 0$ canonically induce a ring homomorphism $t_i^{s.\flat}\colon R_i^{s.\flat} \to R_{i+1}^{s.\flat}$. Thus we have a tower of $\mathbb{F}_p$-algebras $\boldsymbol{R}^\flat = \{R_i^{s.\flat}, t_i^{s.\flat}\}_{i\geq 0}$, which is called the *tilt* of $\boldsymbol{R}$.

One can show that $\boldsymbol{R}^\flat$ is a perfect tower ([INS25, Proposition 3.10 (2)]). We define a refinement of Fontaine's map for perfectoid towers.

Proposition-Definition 3.4 ([HIS26])**.** *For any $i \geq 0$, the canonical projections $R_{i+j} + I_0\widehat{R_\infty} \to (R_{i+j} + I_0\widehat{R_\infty})/I_0\widehat{R_\infty} \cong R_{i+j}/I_0R_{i+j}$ for $j \geq 0$ induce an isomorphism of multiplicative monoids*

$$\varprojlim \left(\cdots \xrightarrow{x\mapsto x^p} R_{i+1} + I_0\widehat{R_\infty} \xrightarrow{x\mapsto x^p} R_i + I_0\widehat{R_\infty} \right) \xrightarrow{\cong} R_i^{s.\flat}.$$

We define i-th Fontaine's (monoidal) map $\sharp^{(i)} : R_i^{s.\flat} \to R_i + I_0\widehat{R_\infty}$ *as the composite of the inverse of the isomorphism above followed by the* 0*-th projection into* $R_i + I_0\widehat{R_\infty}$.

Using the map $\sharp^{(i)}$, we obtain tower-theoretic analogues of Proposition 2.5 and Theorem 2.6:

Proposition 3.5. *Let $I_0^{s.\flat}$ denote the kernel of the 0-th projection $R^{s.\flat} = R_0^{s.\flat} \to R/I_0R$, and let $i \geq 0$.*

(1) *$\sharp^{(i)}$ induces an isomorphism of $\mathbb{F}_p$-algebras*

$$R_i^{s.\flat}/I_0^{s.\flat}R_i^{s.\flat} \xrightarrow{\cong} R_i/I_0R_i. \tag{3.1}$$

(2) *$\sharp^{(i)}$ induces an isomorphism of (possibly) non-unital rings*

$$(R_i^{s.\flat})_{I_0^{s.\flat}\text{-tor}} \xrightarrow{\cong} (R_i)_{I_0\text{-tor}}. \tag{3.2}$$

Note that the isomorphisms (3.1) and (3.2) were already established in [INS25]; Proposition 3.5 simply asserts that they are induced by $\sharp^{(i)}$. We now state our first main theorem.

Theorem 3.6 ([HIS26])**.** *Assume that R is I_0-torsion free, and choose a generator f_0 of I_0. Then the following conditions are equivalent.*

(1) *R_i is completely integrally closed in $R_i[\frac{1}{f_0}]$ for every $i \geq 0$.*
(2) *R_∞ is completely integrally closed in $R_\infty[\frac{1}{f_0}]$.*
(3) *$\widehat{R_\infty}$ is completely integrally closed in $\widehat{R_\infty}[\frac{1}{f_0}]$.*

Moreover, if $\boldsymbol{R}$ satisfies these conditions, then so does the tilt $\boldsymbol{R}^\flat$.

Since the notion of complete integral closedness coincides with that of integral closedness in the Noetherian case, we may expect that Theorem 3.6 can be applied to deduce the normality of small tilts $R_i^{s.\flat}$. As a second main theorem, we give such an example as follows.

Theorem 3.7 ([HIS26])**.** *Let R be an unramified complete regular local ring of mixed characteristic with perfect residue field of characteristic $p > 0$. Let $R \to S$ be a module-finite extension of normal local domains such that $R[\frac{1}{p}] \to S[\frac{1}{p}]$ is étale. For each $n \geq 0$, let S_n be the integral closure of R_n in $(R_n \otimes_R S)[\frac{1}{p}]$. Then there exist a rational $\varepsilon \in (0,1) \cap \mathbb{Q}$, and an integer $N \geq 0$ such that $\{S_n\}_{n \geq N}$ is a perfectoid tower arising from the pair $(S_N, (p^\varepsilon))$.*

Note that the construction and the proof of Theorem 3.7 are heavily based on the work of F. Andreatta [An06]. He had already constructed certain ramified towers as in $\{S_n\}_{n\geq 0}$ appearing in Theorem 3.7 and used them to develop a generalized theory of field of norms. In particular, he made extensive use of ramification theory. Indeed, we choose an integer N as in Theorem 3.7 so that the p-adic valuation of the discriminant of S_N/R_N is sufficiently large. Finally, we obtain the following result as an application of Theorem 3.6.

Corollary 3.8 ([HIS26])**.** *With notation as in Theorem 3.7, $S_n^{s.\flat}$ is a normal ring for every $n \geq N$.*

In fact, the corollary was essentially established in [An06] using a valuation on $S_\infty^\flat$. However, we provide an alternative proof based on the almost purity theorem, one of the most fundamental results in perfectoid theory.

References

[An06] F. Andreatta, *Generalized ring of norms and generalized (ϕ, Γ)-modules*, Ann. Sci. Éc. Norm. Supér. (4) **39**, (2006), 599–647.

[BIM19] B. Bhatt, S. B. Iyengar, L. Ma, *Regular rings and perfect(oid) algebras*, Comm. Algebra **47**(6), (2019), 2367–2383.

[CS24] K. Česnavičius, P. Scholze, *Purity for flat cohomology*, Ann. of Math. (2) **199**(1), (2024), 51–180.

[EHS24] K. Eto, J. Horiuchi, K. Shimomoto, *Some ring-theoretic properties of rings via Frobenius and monoidal maps*, Tokyo J. Mathematics **47**, (2024), 395–409.

[HIS26] K. Hayashi, S. Ishiro, K. Shimomoto, *An application of Fontaine's monoidal map to perfectoid towers*, in preparation.

[INS25] S. Ishiro, K. Nakazato, K. Shimomoto, *Perfectoid towers and their tilts: with an application to the étale cohomology groups of local log-regular rings*, Algebra Number Theory **19**(12), (2025), 2307–2358.

[IS25] S. Ishiro, K. Shimomoto, *δ-rings, perfectoid towers, and lim Cohen-Macaulay sequences*, https://arxiv.org/abs/2509.06527.

[Ish24] R. Ishizuka, *Perfectoid towers generated from prisms*, https://arxiv.org/abs/2409.15785.

[Shi16] K. Shimomoto, *An application of the almost purity theorem to the homological conjectures*, J. Pure Appl. Algebra **220**(2), (2016), 621–632.

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